Janssen, J.; Vandewalle, S.

Convolution SOR waveform relaxation on spatial finite element meshes

This paper investigates the convergence properties of the convolution SOR waveform relaxation method, applied to a system of ordinary differential equations, obtained by spatial finite element discretisation of a linear parabolic initial boundary value problem. We consider both the continuous-time and discrete-time cases and provide a model problem analysis for the one-dimensional heat equation.

1. Introduction

We consider the numerical solution of a linear parabolic initial boundary value problem, spatially discretised by a conforming Galerkin finite element method. This leads to a linear system of d ordinary differential equations (ODEs)

$$B\dot{u}(t) + Au(t) = f(t) \ , \quad u(0) = u_0 \ , \tag{1}$$

with $B = (b_{ij})_{i,j=1}^d$ the symmetric positive definite mass matrix and $A = (a_{ij})_{i,j=1}^d$ the stiffness matrix, [7]. For such systems of ODEs, the standard waveform relaxation method and its multigrid acceleration are investigated in [2, 3]. In this paper, we study the convolution SOR (CSOR) waveform relaxation method, introduced in [6] by Reichelt et al., for general ODE-systems of the form (1). This paper is a summary of [4], where a more detailed study of the method, including proofs and a more extensive reference list, can be found. We begin in §2 by describing the continuous-time and discrete-time CSOR waveform relaxation algorithms. The convergence analysis of these methods is outlined in §3. The paper ends in §4, where theoretical and numerical results are given for the one-dimensional heat equation.

2. CSOR waveform relaxation algorithms

The continuous-time CSOR waveform relaxation algorithm for system (1) computes the new waveform approximation $u_i^{(\nu)}(t)$, $1 \le i \le d$, from the previous approximation in two steps. The first step consists of a Gauss-Seidel like computation of $\hat{u}_i^{(\nu)}(t)$ along a continuous time-interval,

$$\left(b_{ii}\frac{d}{dt} + a_{ii}\right)\hat{u}_i^{(\nu)}(t) = -\sum_{j=1}^{i-1}\left(b_{ij}\frac{d}{dt} + a_{ij}\right)u_j^{(\nu)}(t) - \sum_{j=i+1}^{d}\left(b_{ij}\frac{d}{dt} + a_{ij}\right)u_j^{(\nu-1)}(t) + f_i(t) \ . \tag{2}$$

In the second step the old approximation $u_i^{(\nu-1)}(t)$ is updated. Whereas a standard SOR method would extrapolate the correction by multiplying with an overrelaxation parameter ω, [4], here we convolute the correction with a time-dependent kernel $\Omega(t)$,

$$u_i^{(\nu)}(t) = u_i^{(\nu-1)}(t) + \int_0^t \Omega(t-\tau)\cdot\left(\hat{u}_i^{(\nu)}(\tau) - u_i^{(\nu-1)}(\tau)\right)d\tau \ . \tag{3}$$

The discrete-time CSOR waveform relaxation algorithm is obtained, for example, by applying a linear multistep method, [1], to (2). Hence, the first step becomes

$$\begin{aligned}\sum_{l=0}^{k}\left(\frac{1}{\tau}\alpha_l b_{ii} + \beta_l a_{ii}\right)\hat{u}_i^{(\nu)}[n+l] = &-\sum_{j=1}^{i-1}\sum_{l=0}^{k}\left(\frac{1}{\tau}\alpha_l b_{ij} + \beta_l a_{ij}\right)u_j^{(\nu)}[n+l] \\ &-\sum_{j=i+1}^{d}\sum_{l=0}^{k}\left(\frac{1}{\tau}\alpha_l b_{ij} + \beta_l a_{ij}\right)u_j^{(\nu-1)}[n+l] + \sum_{l=0}^{k}\beta_l f_i[n+l] \ ,\end{aligned} \tag{4}$$

where α_l and β_l are the coefficients of the multistep method, τ is the (constant) step-size, and $u_i^{(\nu)}[n]$ denotes the discrete approximation of $u_i^{(\nu)}(t)$ at $t = n\tau$. In the second step, the continuous-time convolution is replaced by its discrete equivalent,

$$u_i^{(\nu)}[n] = u_i^{(\nu-1)}[n] + \sum_{l=0}^{n} \Omega[n-l] \cdot \left(\hat{u}_i^{(\nu)}[l] - u_i^{(\nu-1)}[l]\right) \ . \tag{5}$$

3. Convergence analysis

By rewriting the CONTINUOUS-TIME iterative scheme (2)–(3) in explicit form, we can derive that the CSOR iteration operator $\mathcal{K}^{CSOR}$ consists of a matrix multiplication and a linear Volterra convolution part,

$$u^{(\nu)}(t) = \mathcal{K}^{CSOR} u^{(\nu-1)}(t) + \varphi(t) = K^{CSOR} u^{(\nu-1)}(t) + \int_0^t k_c^{CSOR}(t-\tau) u^{(\nu-1)}(\tau) d\tau + \varphi(t) \ . \tag{6}$$

Laplace-transformation of (6) leads to $\tilde{u}^{(\nu)}(z) = \mathbf{K^{CSOR}}(z)\tilde{u}^{(\nu-1)}(z) + \tilde{\varphi}(z)$, where $\tilde{u}^{(\nu)}(z)$ denotes the Laplace-transform of the function $u^{(\nu)}(t)$. The continuous-time CSOR symbol equals

$$\mathbf{K^{CSOR}}(z) = \left(z\left(\frac{1}{\tilde{\Omega}(z)} D_B - L_B\right) + \left(\frac{1}{\tilde{\Omega}(z)} D_A - L_A\right)\right)^{-1} \cdot \left(z\left(\frac{1-\tilde{\Omega}(z)}{\tilde{\Omega}(z)} D_B + U_B\right) + \left(\frac{1-\tilde{\Omega}(z)}{\tilde{\Omega}(z)} D_A + U_A\right)\right) \ ,$$

with $B = D_B - L_B - U_B$ and $A = D_A - L_A - U_A$ the standard splittings of B and A in diagonal, lower and upper triangular parts. In terms of this symbol, we can prove the following convergence theorem on infinite time-intervals.

Theorem 1. *Let $\mathcal{K}^{CSOR}$ be an operator in $L_p(0,\infty)$, $1 \le p \le \infty$. Then, $\mathcal{K}^{CSOR}$ is a bounded operator and*

$$\rho\left(\mathcal{K}^{CSOR}\right) = \sup_{\mathrm{Re}(z)\ge 0} \rho\left(\mathbf{K^{CSOR}}(z)\right) = \sup_{\xi \in \mathbb{R}} \rho\left(\mathbf{K^{CSOR}}(i\xi)\right) \ . \tag{7}$$

The following expression for the Laplace-transform of the optimal convolution kernel $\Omega_{opt}(t)$ can be derived, [4].

Theorem 2. *Assume B and A are such that $zB + A$ is consistently ordered,* $\det(zD_B + D_A) \neq 0$*, and the spectrum $\sigma(\mathbf{K^{JAC}}(z)) = \sigma\left((zD_B + D_A)^{-1}(z(L_B + U_B) + (L_A + U_A))\right)$ lies on a line segment $[-\mu_1(z), \mu_1(z)]$ with $\mu_1(z) \in \mathrm{C}$ and $|\mu_1(z)| < 1$. The spectral radius of $\mathbf{K^{CSOR}}(z)$ is then minimised by the unique optimum $\tilde{\Omega}_{opt}(z)$, and is given by*

$$\rho\left(\mathbf{K}^{\mathbf{CSOR},\tilde{\Omega}_{opt}(z)}(z)\right) = |\tilde{\Omega}_{opt}(z) - 1| < 1 \ , \quad \textit{with} \ \ \tilde{\Omega}_{opt}(z) = \frac{2}{1 + \sqrt{1 - \mu_1^2(z)}} \ , \tag{8}$$

where $\sqrt{\cdot}$ denotes the root with the positive real part.

A similar analysis can be done for the DISCRETE-TIME CSOR waveform relaxation method. The operator $\mathcal{K}_\tau^{CSOR}$ turns out to be of discrete convolution type, as we can rewrite (4)–(5) into the following form

$$u^{(\nu)}[n] = \left(\mathcal{K}_\tau^{CSOR} u_\tau^{(\nu-1)}\right)[n] + \varphi[n] = \sum_{i=0}^{n} k[n-l] \ u^{(\nu-1)}[l] + \varphi[n] \ . \tag{9}$$

The discrete-time CSOR symbol $\mathbf{K}_\tau^{\mathbf{CSOR}}(z)$ is obtained by discrete Laplace- or Z-transformation of (9). More precisely, we have $\tilde{u}_\tau^{(\nu)}(z) = \mathbf{K}_\tau^{\mathbf{CSOR}}(z)\tilde{u}_\tau^{(\nu-1)}(z) + \tilde{\varphi}_\tau(z)$ where $\tilde{u}_\tau^{(\nu)}(z)$ denotes the Z-transform of the sequence $u_\tau^{(\nu)} = \{u^{(\nu)}[0], u^{(\nu)}[1], u^{(\nu)}[2], \ldots\}$ and

$$\mathbf{K}_\tau^{\mathbf{CSOR}}(z) = \left(\frac{1}{\tau}\frac{a}{b}(z)\left(\frac{1}{\tilde{\Omega}_\tau(z)} D_B - L_B\right) + \left(\frac{1}{\tilde{\Omega}_\tau(z)} D_A - L_A\right)\right)^{-1} \left(\frac{1}{\tau}\frac{a}{b}(z)\left(\frac{1-\tilde{\Omega}_\tau(z)}{\tilde{\Omega}_\tau(z)} D_B + U_B\right) + \left(\frac{1-\tilde{\Omega}_\tau(z)}{\tilde{\Omega}_\tau(z)} D_A + U_A\right)\right) \ ,$$

with $a(z) = \sum_{j=0}^{k} \alpha_j z^j$ and $b(z) = \sum_{j=0}^{k} \beta_j z^j$ the characteristic polynomials of the multistep method. We then obtain the following discrete-time equivalents of Theorems 1 and 2.

Theorem 3. *Let $\mathcal{K}_\tau^{CSOR}$ be an operator in $l_p(\infty)$, $1 \le p \le \infty$. Then $\mathcal{K}_\tau^{CSOR}$ is a bounded operator and*

$$\rho\left(\mathcal{K}_\tau^{CSOR}\right) = \max_{|z|\ge 1} \rho\left(\mathbf{K}_\tau^{\mathbf{CSOR}}(z)\right) = \max_{|z|=1} \rho\left(\mathbf{K}_\tau^{\mathbf{CSOR}}(z)\right) \ . \tag{10}$$

Theorem 4. *Assume B and A are such that $\frac{1}{\tau}\frac{a}{b}(z)B + A$ is consistently ordered, $\det(\frac{1}{\tau}\frac{a}{b}(z)D_B + D_A) \neq 0$, and the spectrum $\sigma(\mathbf{K}_\tau^{\mathbf{JAC}}(z)) = \sigma(\mathbf{K}^{\mathbf{JAC}}(\frac{1}{\tau}\frac{a}{b}(z)))$ lies on a line segment $[-(\mu_1)_\tau(z), (\mu_1)_\tau(z)]$ with $(\mu_1)_\tau(z) \in \mathbf{C}$ and $|(\mu_1)_\tau(z)| < 1$. The spectral radius of $\mathbf{K}_\tau^{\mathbf{CSOR}}(z)$ is then minimised by the unique optimum $(\tilde{\Omega}_{opt})_\tau(z)$, and is given by*

$$\rho(\mathbf{K}_\tau^{\mathbf{CSOR},(\tilde{\Omega}_{opt})_\tau(z)}(z)) = |(\tilde{\Omega}_{opt})_\tau(z) - 1| < 1 \quad , \quad \textit{with} \quad (\tilde{\Omega}_{opt})_\tau(z) = \frac{2}{1 + \sqrt{1 - (\mu_1)_\tau^2(z)}} \ , \tag{11}$$

where $\sqrt{\cdot}$ denotes the root with the positive real part.

By comparison of (8) and (11), we observe that $(\tilde{\Omega}_{opt})_\tau(z) = \tilde{\Omega}_{opt}\left(\frac{1}{\tau}\frac{a}{b}(z)\right)$. Consequently, in the optimal case, (10) can be rewritten as

$$\rho\left(\mathcal{K}_\tau^{CSOR,(\Omega_{opt})_\tau}\right) = \sup\left\{ \rho\left(\mathbf{K}^{\mathbf{CSOR},\tilde{\Omega}_{opt}(z)}(z)\right) \;\middle|\; \tau z \in \mathbf{C} \setminus intS \right\} = \sup_{z \in \frac{1}{\tau}\partial S} \rho\left(\mathbf{K}^{\mathbf{CSOR},\tilde{\Omega}_{opt}(z)}(z)\right) \ , \tag{12}$$

where S denotes the stability region of the multistep method.

4. Model problem analysis

Consider the one-dimensional heat equation

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta_1 \mathbf{u} = 0 \ , \quad x \in [0,1] \ , \quad t \in [0,1] \ , \tag{13}$$

with homogeneous Dirichlet boundary conditions and a given initial condition. Let the equation be discretised using linear finite element basis functions on a grid with mesh-size h, i.e., $\Omega_h = \{x_i = ih \mid 0 \le i \le 1/h\}$. This leads to a system of ODEs (1) with (in stencil notation) $B = [\frac{h}{6} \ \frac{4h}{6} \ \frac{h}{6}]$ and $A = [\frac{-1}{h} \ \frac{2}{h} \ \frac{-1}{h}]$.

Theorem 5. *Consider $\mathcal{K}^{CSOR,\Omega_{opt}(t)}$ as an operator in $L_p(0,\infty)$, $1 \le p \le \infty$, for solving (13), discretised using linear finite element basis functions. Then, for small h, we have that*

$$\rho\left(\mathcal{K}^{CSOR,\Omega_{opt}(t)}\right) \approx 1 - 2\pi h \ . \tag{14}$$

The proof of Theorem 5 is based on the observation that the maximum of $\rho(\mathbf{K}^{\mathbf{CSOR},\tilde{\Omega}_{opt}(z)}(z))$ along the imaginary axis is attained at the origin. Hence, $\rho(\mathcal{K}^{CSOR,\Omega_{opt}(t)})$ equals the spectral radius of the algebraic SOR method for matrix A with optimal overrelaxation parameter $\tilde{\Omega}_{opt}(0)$, which is well-known to be $1 - 2\pi h$ for small h, [8].

Table 1 presents some averaged convergence factors obtained with an implementation of the discrete-time convolution SOR waveform relaxation method with optimal convolution kernel, for model problem (13). We used the Crank-Nicolson (CN) method and the backward differentiation (BDF) formulae of order 1, 3 and 5, with time-step $\tau = 1/100$. It is well-known for waveform relaxation methods that the numerical results, although obtained on finite time-intervals, match the infinite time-interval theoretical analysis, see e.g. [5] for a theoretical explanation based on the pseudospectra of the relevant operators. Indeed, the observed convergence factors in Table 1 are in close correspondence with the theoretical result (14). Also, the results of Table 1 show that, for a fixed value of h, the observed convergence factors are more or less independent of the chosen time-discretisation method. An explanation of this behaviour can be found in Figure 1, where we visualise the application of formula (12) for model problem (13) by means of a so-called spectral picture: we display contour lines of $\rho(\mathbf{K}^{\mathbf{CSOR},\tilde{\Omega}_{opt}(z)}(z))$ (for values

h	1/8	1/16	1/32	1/64
CN	0.388	0.637	0.802	0.899
BDF(1)	0.389	0.637	0.802	0.896
BDF(3)	0.378	0.630	0.797	0.894
BDF(5)	0.377	0.629	0.797	0.894
$1 - 2\pi h$	0.215	0.607	0.804	0.902

Table 1: Observed convergence factors for (13) – optimal CSOR waveform relaxation – $\tau = 1/100$.

0.6, 0.7, 0.8 and 0.9), together with (parts of) the scaled stability region boundaries $\frac{1}{\tau}\partial S$ of the CN method and the BDF methods of order 1, 3 and 5.

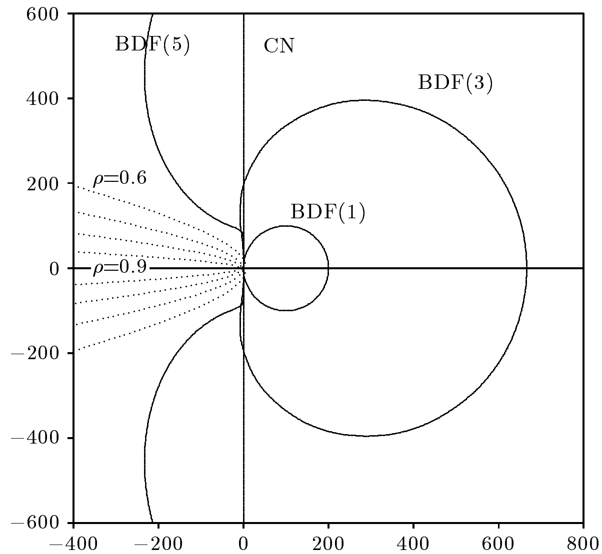

Figure 1: Spectral picture for (13) – optimal CSOR waveform relaxation – $h = 1/16$, $\tau = 1/100$.

Acknowledgements

This text presents research results of the Belgian Incentive Programme 'Information Technology'—Computer Science of the Future (IT/IF/5), initiated by the Belgian State—Prime Minister's Service— Federal Office for Scientific, Technical and Cultural Affairs. The scientific responsibility is assumed by its authors. The work of the second author was supported in part by the NSF under Cooperative Agreement No. CCR-9120008.

5. References

1 Hairer, E., Wanner, G.: Solving Ordinary Differential Equations II. Volume 14 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1991.
2 Janssen, J., Vandewalle, S.: Multigrid waveform relaxation on spatial finite element meshes: the continuous-time case. SIAM J. Num. Anal. **33**(2), April 1996 (to appear).
3 Janssen, J., Vandewalle, S.: Multigrid waveform relaxation on spatial finite element meshes: the discrete-time case. SIAM J. Sci. Comput. **17**(1), January 1996 (to appear).
4 Janssen, J., Vandewalle, S.: On SOR waveform relaxation methods. Techn. Report CRPC, California Institute of Technology, Pasadena, California, 1995 (in preparation).
5 Lumsdaine, A., Wu, D.: Spectra and pseudospectra of waveform relaxation operators. Techn. Report CSE-TR-95-14, Department of Computer Science and Engineering, University of Notre Dame, Notre Dame, Indiana, June 1995.
6 Reichelt, M. W., White J. K., Allen, J.: Optimal convolution SOR acceleration of waveform relaxation with application to parallel simulation of semiconductor devices. SIAM J. Sci. Comput., **16**(5), September 1995 (to appear).
7 Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Volume 1054 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1984.
8 Young, D. M.: Iterative Solution of Large Linear Systems. Academic Press, New York, 1971.

Addresses: Janssen Jan, Katholieke Universiteit Leuven, Department of Computer Science, Celestijnenlaan 200A, B-3001 Heverlee, Belgium.

Vandewalle Stefan, California Institute of Technology, Applied Mathematics 217-50, Pasadena, CA 91125, USA.